

Tip Splittings and Phase Transitions in the Dielectric Breakdown Model: Mapping to the DLA Model.

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We show that the fractal growth described by the dielectric breakdown model exhibits a phase transition in the multifractal spectrum of the growth measure. The transition takes place because the tip-splitting of branches forms a fixed angle. This angle is η dependent but it can be rescaled onto an “effectively” universal angle of the DLA branching process. We derive an analytic rescaling relation which is in agreement with numerical simulations. The dimension of the clusters decreases linearly with the angle and the growth becomes non-fractal at an angle close to 74° (which corresponds to $\eta = 4.0 \pm 0.3$).

Fractal growth and patterns are common phenomena of nature. The prototype model for a mathematical description of fractal growth is the diffusion-limited aggregation (DLA) model [1]. The growth in this model is determined by the electric field (called the harmonic measure) around the emerging fractal cluster. The harmonic measure possesses multifractal scaling properties and recently, a new insight into the behavior of this measure was presented by applying the method of infinitely convoluted conformal mappings [2]. In particular it has been demonstrated that there exists a singular behavior in the multifractal spectrum and this singularity is a signature of a phase transition in thermodynamics formalism of the spectrum [3]. The phase transition in the DLA cluster occurs at a specific moment q_c of probabilities of the growth. It has further been demonstrated that the geometrical properties of the DLA cluster, which gives rise to this transition, is the existence of a specific branching angle for each new offspring on this cluster [4]. For DLA this critical branching angle was found to be around 27° . Another approach which defines a characteristic angle in the DLA process, is to consider the stability of a finger growing in a wedge [5].

The DLA model has been nicely generalized to the dielectric breakdown model (DBM) [6] where the growth probabilities at a specific site of the cluster is determined by the value of the electric field (or the harmonic measure) raised to a power η , i.e. the growth measure at the interface follows, $\rho_\eta(s)ds \sim |E(s)|^\eta ds$ where DLA corresponds to the case $\eta = 1$. With varying values of η clusters of different geometry are grown each with their own characteristic properties of the multifractal spectrum. These properties have not been outlined before and it is the purpose of this letter to examine in details possible critical points and phase transitions in the thermodynamic formalism of the harmonic measure for the dielectric breakdown model, at varying values of η .

The structure of the clusters of the DBM model emerges from an on-going proliferation and screening, hence stagnation, of branches. In particular will the protruding branches create fjords in which the harmonic measure decreases rapidly compared to what happens around the tips. The multifractal properties are best studied using the recently proposed model of iterated

conformal maps [2], since the deep fjords, numerically, are invisible to the original approach where random walkers are used as probes, see [3]. The model is based on compositions of simple conformal maps $\phi_{\lambda,\theta}$ which take the exterior of the unit circle to its exterior, except for a little bump at $e^{i\theta}$ of linear size proportional to $\sqrt{\lambda}$. We shall here use the mapping introduced in [2] which produces two square root singularities which we refer to as the branch cuts, and the tip of the bump which we refer to as the micro tip. The composition of these mappings is analog to the aggregation of random walkers in the off-lattice DLA model. The dynamics is given by

$$\Phi^{(n)}(w) = \Phi^{(n-1)}(\phi_{\lambda_n, \theta_n}(w)) . \quad (1)$$

The size of the n 'th bump is controlled by the parameter λ_n and in order to achieve particles of fixed size we have that, to first order,

$$\lambda_n = \frac{\lambda_0}{|\Phi^{(n-1)'}(e^{i\theta_n})|^2} . \quad (2)$$

The growth probability $\rho_1(s)$ at the interface of a DLA cluster of size n is, in the electrostatic picture, proportional to the electric field

$$\rho_1(s)ds \sim |E(s)|ds \sim \frac{ds}{|\Phi'|} . \quad (3)$$

In this case the measure on the unit circle is uniform. When the electric field is raised to the power η , the measure is no longer uniform,

$$\rho_\eta(\theta)d\theta \sim \rho_\eta(s(\theta)) \left| \frac{ds}{d\theta} \right| d\theta \sim \frac{|E|^\eta}{|E|} d\theta \sim |\Phi'(e^{i\theta})|^{1-\eta} d\theta . \quad (4)$$

Numerically we use the Monte Carlo technique introduced in [9] in order to choose θ according to the distribution ρ_η . We vary the number of iterations T such that for $\eta = 1.25$, $T = 50$ and for $\eta = 4$, $T = 400$.

Below we consider both the growth measure and the harmonic measure. The growth measure is used to derive the multifractal properties whereas the harmonic measure is used to determine the physical properties. Let us emphasize that during the growth we always use the growth measure. First, we consider the growth measure

and the phase transition in the corresponding multifractal spectrum.

The moments of the growth probability scales with characteristic exponents, the generalized dimensions,

$$\int \rho^q(s) ds \sim (1/R)^{(q-1)D_q} \sim n^{(1-q)D_q/D}, \quad (5)$$

where n is the number of particles and R the linear size of the cluster.

Numerically we approximate the integral on the left hand side by the sum of the field evaluated along the micro tips of the bumps produced by the bump mappings. The field in DLA will for clusters of size 20000 assume values below 10^{-20} and it follows from (3) that it is impossible with the numerical precision on the unit circle ($\Delta\theta \simeq 10^{-16}$) to maintain a reasonable resolution in the physical space $\Delta s = |E|\Delta\theta \simeq 10^4$. We therefore use the resolution increasing approach introduced in [3] where one keeps track on the dynamics of the branch-cuts.

An easy way to see the existence of a phase transition in the multifractal spectrum is to look at the distribution of ρ_η sampled along the tips of the bumps. The distribution will for the smallest values (below some cutoff value c) of $\rho_\eta < c$ scale with a characteristic exponent $\frac{1-\beta}{\beta}$. The value of β is calculated by reordering the N computed values of ρ_η in ascending order. In other words, we write them as a sequence $\{\rho_\eta(i)\}_{i \in I}$ where I is an ordering of the indices such that $\rho_\eta(i) \leq \rho_\eta(j)$ if $i < j$. We treat the discrete index i/N as a continuous index $0 \leq x \leq 1$ and therefore ρ_η as a non-decreasing function of x ,

$$\rho_\eta \equiv f(x). \quad (6)$$

Numerically we find that the function $f(x)$ obeys a power law with an exponent β for values of $x \ll 1$. From $f(x)$ we calculate the distribution of $p(\rho_\eta)$ by the transformation formula

$$p(\rho_\eta) \sim \int \delta(\rho_\eta - f(x)) dx = \frac{1}{|f'(x(\rho_\eta))|} \sim \rho_\eta^{\frac{1-\beta}{\beta}}. \quad (7)$$

With the use of this distribution the moment integral (5) is rewritten as

$$\begin{aligned} \int_0^L \rho_\eta^q ds &= \int_0^\infty \rho_\eta^q p(\rho_\eta) d\rho_\eta \\ &= k \int_0^c \rho_\eta^q \rho_\eta^{\frac{1-\beta}{\beta}} d\rho_\eta + \int_c^\infty \rho_\eta^q p(\rho_\eta) d\rho_\eta, \end{aligned} \quad (8)$$

where k is some normalization constant. The left integral in the final expression diverges whenever

$$q \leq q_c = -\frac{1}{\beta}. \quad (9)$$

The phase transition in the multifractal spectrum of the growth measure takes place for the value $q = q_c$. We shall

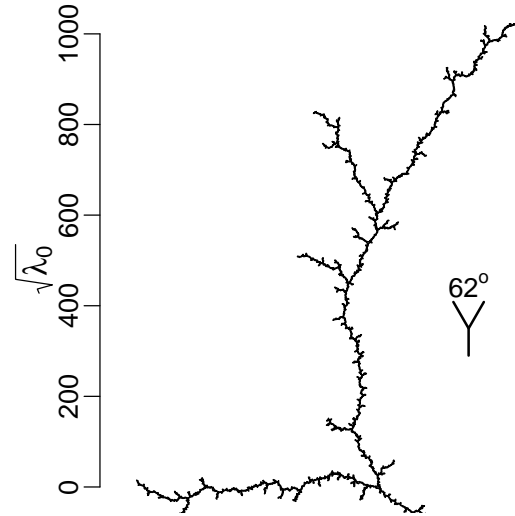


FIG. 1: Part of a cluster grown with $\eta = 3$. The wedge structure at the bottom of the fjords is clearly seen and the opening angle observed along the aggregate is close to the angle predicted in (13) and shown on the figure.

now argue that the value of β , and therefore q_c , is independent of the value of η . In other words the phase transition in the multifractal spectrum of the growth measure is universal.

In [4] it was shown that the angle defining the branch splitting deep inside the fjords of DLA was given by a characteristic angle $\gamma_c(1) = \gamma_c(\eta = 1)$. This characteristic angle was also identified as the reason for the phase transition in the multifractal spectrum of the harmonic measure of DLA. The electric field along the branches in a wedge with opening angle γ scales like

$$\rho_1(x) \propto |E(x)| \sim x^{\frac{\pi}{\gamma}-1} \quad (10)$$

and therefore in DBM the growth probability inside this wedge is given by

$$\rho_\eta(x) \propto |E(x)|^\eta \sim x^{\eta(\frac{\pi}{\gamma}-1)}. \quad (11)$$

When the exponent η is introduced, the growth probability (11) inside the wedge is changed and therefore we see a corresponding change in the geometry, such that e.g. when $\eta > 1$ the field inside a wedge with opening angle $\gamma_c(1)$ is effectively similar to that of a wedge with a smaller opening angle.

The wedge structure at the bottom of the fjords of DLA is not affected by a change in the value of η see Fig. 1, the physical angle, however, is changed such that the effective angle remains fixed and equal to the angle observed in DLA. Generally we therefore have for the angle observed physically that $\gamma_c(\eta > 1) > \gamma_c(1)$ and $\gamma_c(\eta < 1) < \gamma_c(1)$. The angle for a given value of η is

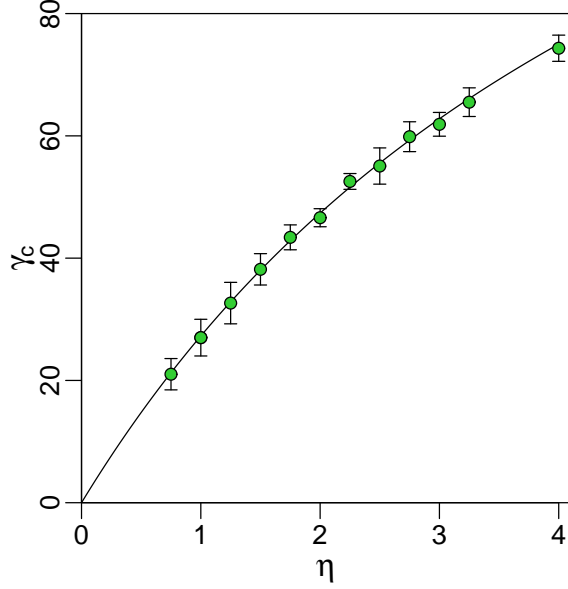


FIG. 2: The critical angle as function of η . The line represents the critical angle obtained from the analytical expression (13). Each dot represent the numerical average of 4-15 clusters of size $n = 18000$. The standard deviation of the individual points assumes values in the range between 1.5° and 3.4° .

found by comparing the exponents in (11) and (10)

$$\left(\frac{\pi}{\gamma_c(\eta)} - 1\right)\eta = \frac{\pi}{\gamma_c(1)} - 1 \quad (12)$$

or

$$\gamma_c(\eta) = \frac{\pi\eta}{\frac{\pi}{\gamma_c(1)} - 1 + \eta} . \quad (13)$$

Fig. 2 shows together with numerical predictions (see below) how the critical angle varies with η . The value of $\gamma_c(1)$ in (13) and used in the figure is determined numerically from 15 DLA clusters of size $n=20000$, $\gamma_c(1) = 27^\circ \pm 3^\circ$.

The distribution of ρ_η inside a wedge with opening angle γ follows from (11), with $\alpha = \pi/\gamma$,

$$p(\rho_\eta) \sim \frac{1}{(x(\rho_\eta))^{\eta(\alpha-1)-1}} \sim \rho_\eta^{\frac{1-\eta(\alpha-1)}{\eta(\alpha-1)}} , \quad (14)$$

and if we compare the exponent with that of (7) and insert the critical angle from (12) we find that

$$\beta = \frac{\pi}{\gamma_c(1)} - 1 . \quad (15)$$

Therefore β in expression (9) is independent of η .

One way, numerically, to calculate the critical angles shown in Fig. 2 is first to locate the regions where the distribution in (7) scales and afterwards perform a direct measurement in these regions. Such measurements are most likely rather inaccurate and therefore we turn to

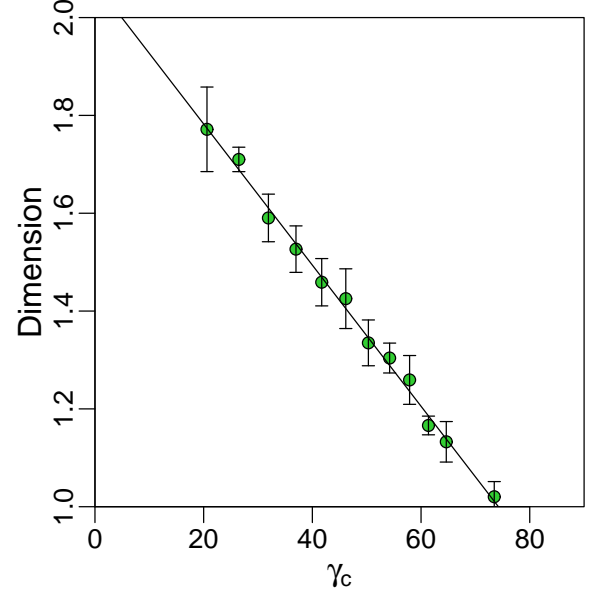


FIG. 3: The estimated dimension plotted versus the critical angle γ_c , see (13). The range of the critical angle corresponds to values of η between 0.75 and 3.5. The linear fit of the points intersects the line $D=1$ at an angle $\gamma_c \approx 74^\circ$.

(10). During the growth we apply the exponent of η as usual but once a cluster is grown we consider the harmonic measure ρ_1 only. Similarly, as above, we find that inside a wedge

$$p(\rho_1) \sim \rho_1^{\frac{1-(\alpha-1)}{\alpha-1}} . \quad (16)$$

The exponent is compared to the one we calculate numerically from (7) and from this we find the critical angle as function of β . Note that when we consider the harmonic measure and not the growth measure, β is a nontrivial function of η and the critical angle is given by

$$\gamma_c(\eta) = \frac{\pi}{1 + \beta(\eta)} . \quad (17)$$

Another interesting observation is that the dimension seems to depend linearly on the critical angle as shown in Fig. 3. Note that the linear fit intersects the line of value equal one before the angle reaches 180 degrees, and therefore the growth is only fractal for η below a finite value.

The point of intersection is found at the angle $\gamma_c = 74^\circ$ and using (13) we find that

$$\eta = 4.0 \pm 0.3 , \quad (18)$$

in agreement with the results obtained in [7] and [9]. To sum up on these results, we rewrite the relation between the dimension and the critical angle in terms of the angles γ_2 and γ_1 at which the growth becomes two- and one-dimensional respectively

$$D(\gamma_c) = 1 + \frac{\gamma_1 - \gamma_c}{\gamma_1 - \gamma_2} , \quad \gamma_2 \leq \gamma_c \leq \gamma_1 . \quad (19)$$

The dimension can also by (13) be written in terms of η and in this case the dependence will no longer be linear but be of a form similar to that observed in [7]. Due to the finite size of the bumps used in the growth we do not observe, as $\eta \rightarrow 0^+$, that the critical angle of the fjords vanishes. The bumps will fill up the fjords, and the growth become two dimensional for a non-vanishing value of η and $\gamma_2 \approx 5^\circ$. The fill up problem is also the reason why we have not been able to present data points for values of η below .75, because the deep fjords are filled up.

In conclusion, we have studied the critical properties of the growth of clusters in the dielectric breakdown model. In particular, we have focused on the branching process and have found that the branching, on the average, oc-

curs at a fixed angle which depends on the value of the characteristic parameter η of the DBM model. The size of the angle in turn determines the phase transition point of the growth measure. We have derived an analytic expression for the variation of the branching angle with η and found excellent agreement with numerical data. Further, we have found a linear dependence of the dimension of the cluster with the value of critical angle. This linear dependence results in a prediction of the branching angle at the point where the growth becomes one-dimensional. It is found to be $\gamma_c \approx 74^\circ$ corresponding to $\eta_c \approx 4$ in agreement with a result obtained in [7] and [9]. We are in the process of developing a scaling theory for the linear dependence of the dimension versus angle.

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